# Ch. 4 Summary

## Num. Differentiation

- Interpolate using Lagrange interpolant
- $(n+1)$ points $x_0, x_1, \ldots, x_n$
- Derive a $(n+1)$ pt formula, $O(h^n)$
  $h = x_{j+1} - x_j$
- Unstable

## Num. Integration

- Trapezoid / Comp. Trapezoid $O(h^2)$
- Simpson's / Comp. Simpson's $O(h^4)$
- Stable
- $\underbrace{\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} \overset{\text{weight}}{c_i}\, \overset{\text{nodes}}{f(x_j)}}_{\text{quadratic rule}}$

# Ch. 5 - Initial Value Problems (IVP) for ODEs

## 5.1 - IVP, well-posedness

Def: An IVP is of the form

$$dy/dt = f(y,t), \quad a \le t \le b, \quad \underbrace{y(a) = \alpha}_{\text{Initial condition}}$$

Ex) $dy/dt = 2y + e^{-t} + y^2/t$

First questions to ask before doing an IVP (solving)

1) Does I.V.P have a solution? Existence
2) Is this solution unique? Uniqueness
3) Is the solution STABLE? Stability

Checking change of position; is it volatile?

Small perterbation of the initial condition and/or the equation imply small perturbation of solution

Coding exam - 1 method out of 3
Midterm - Ch. 3-4, 1 hr written test

# ~~Section 5.2:~~ Section 5.1 cont'd

How do we assure Existence, Uniqueness, and Stability?

How to ensure well-posedness of the IVP?

**Definition:** Lipschitz's condition

$f(t,y)$ satisfies a Lipschitz condition if
$\forall\ (t,y_1),\ (t,y_2)$ in the domain of $f$
$$|f(t,y_1) - f(t,y_2)| \le L\,|y_1 - y_2|$$
Lipschitz constant

**Interpretation:**

$$\underbrace{\left|\frac{f(t,y_1) - f(t,y_2)}{y_1 - y_2}\right|}_{\approx \left|\partial f/\partial y\right|} \le L$$

partial derivative of $f$ with respect to $y$ OR smooth variation of $f$ w/ respect to $y$

**Theorem:** If $f \in C(D)$, $D = \{(t,y) \,/\, a \le t \le b,\ y \in \mathbb{R}\}$
domain

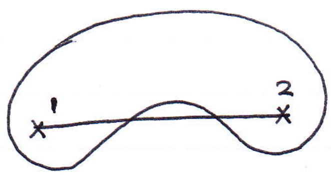
It satisfies a Lipschitz condition
$\Rightarrow$ the IVP is well-posed

**Theorem:** (Lipschitz Condition) If $f$ is defined on a convex set

If there exists $L \geq 0$ / $\left| \partial f / \partial y \right| \leq L$

$\Rightarrow$ $f$ satisfies a Lipschitz condition

$\Rightarrow$ IVP is well-posed

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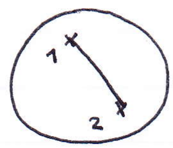
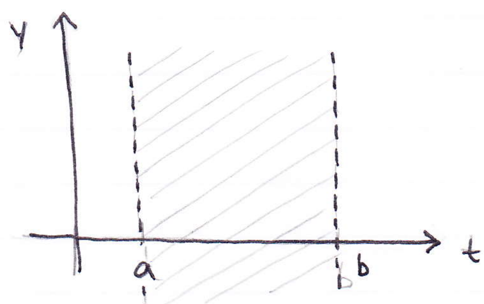



Included

Convex: when line that connects 2 points in the domain is included in the domain




Domain is 2D

Image of Theorem 1

Ex) $y' = y\cos t \qquad 0 \leq t \leq 1 \qquad y(0) = 1$ Well-posed?

1) Define $f(t, y) = y\cos t$

Option 1) $|f(t, y_1) - f(t, y_2)| = |y_1 \cos t - y_2 \cos t|$

$= |\cos t|\,|y_1 - y_2| \leq \, ? \, |y_1 - y_2|$

Bounds of $\cos = [-1, 1]$ so

$|\cos t|\,|y_1 - y_2| \leq 1 \times |y_1 - y_2| \quad \forall t \in [0, 1]$

Option 2) $\partial f / \partial y = \cos t, \quad |\partial f / \partial y| = |\cos t|$

$\leq 1 \quad \forall t \in [0, 1]$

Well-posed for $L = 1$ ($1 \geq 0$, so well-posed)

# Section 5.2 - Euler's Method

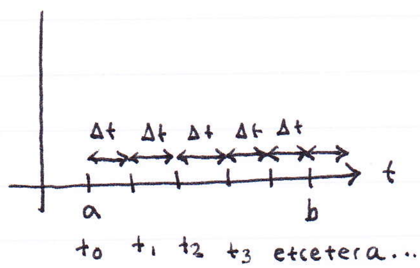
Euler's method is a <u>fundamental</u> numerical scheme (or method) to solve IVPs.

Solve numerically:

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

1) Discretize the time:

Sample $[a, b]$ using Time Step $\Delta t = \frac{b-a}{N}$, $N$ = number of sub-intervals




for $j = 0, \dots, n$

$t_j = a + j\Delta t$

2) Discretize $y'(t)$ $\Rightarrow$ numerical differentiation
$(n+1)$ point formula

Easiest one: forward difference

$$y'(t) \approx \frac{y(t+\Delta t) - y(t)}{\Delta t}$$

$$y'(t_j) = f(t_j, y(t_j)) \quad \forall j = 0, \dots, N$$

$$\frac{y(t_j + \Delta t) - y(t_j)}{\Delta t} \approx f(t_j, y(t_j))$$

$\star$ $t_j + \Delta t = t_{j+1}$

Notation:

$W_j$ = the approximation of $y(t_j)$

$\Rightarrow$ We set $\frac{W_{j+1} - W_j}{\Delta t} = f(t_j, W_j) \Leftrightarrow$

Euler's Method

$\star$ $W_0 = \alpha$, $\Delta t = \frac{b-a}{N}$

$W_{j+1} = W_j + \Delta t f(t_j, W_j)$

$(y - e^t)$

Ex) $y'(t) = y(t) - e^t$, $0 \le t \le 1$, $y(0) = 2$

1) Well-posed?

2) Find exact solution

3) Apply Euler's method for $\Delta t = 0.25$

~~$\partial f / \partial y$~~ 3) If $\Delta t = 0.25$, then $N = 1/\Delta t = 4$

$W_0 = 2$ Initial condition

Define function $f(t, y) = y - e^t$

Euler's method defined by $w_{j+1} = w_j + \Delta t\, f(t_j, w_j)$

$j = 0, 1, 2, 3, 4$

$W_1 = W_0 + 0.25\,(f(t_0, w_0)) = 2 + 0.25 f(0, 2)$

$= 2 + 0.25\,(1) = 2.25$

$W_2 = w_1 + 0.25\,(f(t_1, w_1)) = 2.25 + 0.25 f(0.25, 2.25)$

$= 2 + 0.25\,(\cancel{2.2}\,2.25 - e^{0.25})$

$= 2.4915$

and so on...

$W_3 = 2.7022$ $\quad W_4 = 2.8485$

Comparisons

| | |
|---|---|
| $w_1 = 2.25$ | $y(t_1) = 2.2470$ |
| $w_2 = 2.4915$ | $y(t_2) = 2.4731$ |
| $w_3 = 2.7022$ | $y(t_3) = 2.6463$ |
| $w_4 = 2.8485$ | $y(t_4) = 2.7183$ |

$\downarrow$ Error Increases, Why?

## Cont'd From Last Time

| t | Approx | Exact |
|---|---|---|
| 0 | 2 | 2.0 |
| 0.25 | 2.25 | 2.247 |
| 0.5 | 2.49 | 2.4731 |
| 0.75 | 2.70 | 2.6463 |

**Euler's Method;** $t_j = a + j\Delta t$, $j = 0, \ldots, N$

$\Delta t = b - a / N$

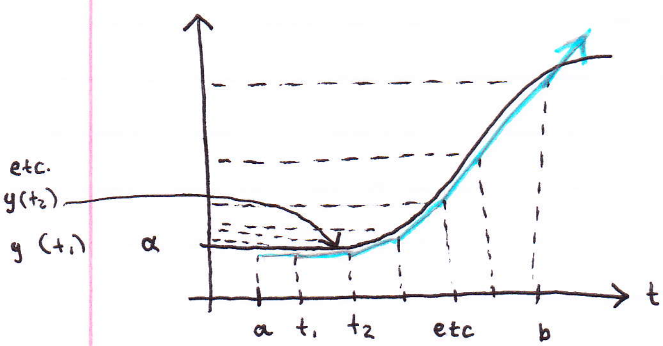
$w_0 = \alpha$

$w_{j+1} = w_j + \Delta t\,(f(t_j, w_j))$

<u>Conclusion:</u> $\underset{\text{error}}{|y(t_j) - w_j|}$ increases as $j$ increases

In other words, error accumulates over time

<u>Geometrical Interpretation:</u>




$y' = f(t, y)$

$w_{t_{j+1}} = w_j + \Delta t\, f(t_j, w_j)$

$\downarrow$

$Z(t) = w_j + t\, f(t_j, w_j)$ line

$\approx \underbrace{y(t_j) + t\, y'(t_j)}$

looks like TANGENT line of $y$ at $t_j$

* Error is Linear * in $\Delta t$

~~Error Bound~~

## Error Bound for Euler's Method:

Theorem:
- Suppose $f \in C(D)$, satisfies a Lipschitz condition over $D$
- Suppose, there exists $M > 0$ / $\left| \frac{\partial^2 f}{\partial t^2}(t, y) \right| \leq M$ for $M$ $\forall \in [a, b]$

$$|y(t_j) - w_j| \leq \frac{\Delta t M}{2L} \left( \exp(L(t_j - a)) - 1 \right)$$

Summary: $|y(t_j) - w_j| \leq C(\Delta t)$

rate of converge $O(\Delta t)$

## How to Improve the Method?

Remark:
1) Approximate model (the IVP)
2) Approximate solution (solution of approx model)
3) What we did - compare exact solution of the "exact" model with ___ approx. _____ approx model

How to compare models?

LOCAL TRUNCATION ERROR

Approximate Model:

$$w_{j+1} = w_j + \Delta t \underbrace{(\phi(t_j, w_j))}_{\text{function}}$$

For Euler's method, $\phi = f$

Compare IVP and approx. models

$y$ solution of $y' = f(t, y)$

$$y'(t_j) = f(t_j, y(t_j))$$

If models are close, then

$$\underbrace{\frac{y(t_{j+1}) - y(t_j)}{\Delta t} - \phi(t_j, y(t_j))}_{\text{local truncation error}} \approx 0$$

$$\tau_{j+1}(\Delta t) = \frac{y(t_{j+1}) - y(t_j)}{\Delta t} - \phi(t_j, y(t_j))$$

General local truncation error $T_j$ is error

For Euler's :

$$\tau_{j+1}(\Delta t) = \frac{y(t_{j+1}) - y(t_j)}{\Delta t} - y'(t_j) = O(\Delta t)$$

2-pt formula

$(n+1)$ pts $= O(h^n)$

Other Derivation: Taylor Series rather than Lagrange Interpolant

$$y(t_j + \Delta t) = \underbrace{y(t_j) + y'(t_j)\Delta t}_{\text{polynomial}} + \underbrace{\frac{y''(\xi(t))}{2!}\Delta t^2}_{\text{error term}}$$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(\cancel{x_0})(\xi(x))}{2!}(x - x_0)^2$$

$$y = f, \quad x = t_j + \Delta t, \quad x_0 = t_j$$

$$y(t_j + \Delta t) = y(t_j) + y'(t_j)\Delta t + \frac{y''(\xi(t))}{2!}\Delta t^2$$

$$\frac{y(t_{j+1}) - y(t_j)}{\Delta t} - y'(t_j) = \frac{y''(\xi(t))\Delta t}{2}$$

same result, still $O(\Delta t)$

How to Improve this Method?

→ Use Taylor's Formula with MORE terms $(n > 1)$

$$y(t_{j+1}) = y(t_j) + y'(t_j)\Delta t + \frac{y''(t_j)}{2!}\Delta t^2 + \dots + \frac{y^{(n)}(t_j)\Delta t^n}{n!} + \text{error}$$

$$\text{error} = \frac{y^{(n+1)}(\xi(t))\Delta t^{n+1}}{(n+1)!}$$

$$\frac{y(t_{j+1}) - y(t_j)}{\Delta t} = y'(t_j) + \frac{y''(t_j)}{2!}\Delta t + \dots + \frac{y^{n}(t_j)}{n!}\Delta t^{n-1} + \frac{y^{(n+1)}(\xi(t))\Delta t^n}{(n+1)!}$$

→

$$\frac{y(t_{j+1}) - y(t_j)}{\Delta t} = f(t_j, y(t_j)) + \frac{f'(t_j, y(t_j))\Delta t}{2!} + \dots + \frac{f^{n+1}(\xi(t))\Delta t^{n}}{(n+1)!}$$

From Taylor's Series

$$\frac{y(t_{j+1}) - y(t_j)}{\Delta t} - \underbrace{\left[ f(t_j, y(t_j)) + \frac{\Delta t}{2} f(t, y(t_j)) + \dots + \frac{f^{(n-1)}(t_j, y(t_j))\Delta t}{(n!)} \right]^{n-1}}_{\phi(t_j,\, y(t_j))} = O(\Delta t^{n}$$

$$T^{(n)}(t_j, w_j) = f(t_j, w_j) + \frac{\Delta t}{2} f'(t_j, w_j) + \dots + \frac{(\Delta t)^{n-1}}{n!} f^{(n-1)}(t_j, w_j)$$

with error bound $O(\Delta t^n)$

Ex) $y' = y - t^2 + 1$, $0 \le t \le 2$, $y(0) = 0.5$. Apply second Taylor's method for $N = 0$

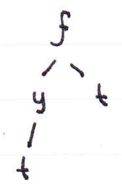
- $\Delta t = 2 - 0/10 = 0.2$
- $f(y, t) = y - t^2 + 1$
  $\hookrightarrow f(y(t), t)$

The 2nd taylor's method is given by

$$\begin{cases} w_0 = 0 \\ w_{j+1} = w_j + \Delta t\, f(t_j, w_j) + \frac{\Delta t}{2} f'(t_j, w_j) \end{cases}$$

$f'(y(t), t) = y'(t)\, \partial f/\partial y + \partial f/\partial t$

Use chain rule




then here $f'(y(t)t)$ becomes $f(y't)$ * $y' = y - t^2 + 1$

$f'(y(t), t) = f(y, t) \times 1 + (-2t)$

$= (y - t^2 + 1) - 2t$

so $w_{j+1} = w_j + 0.2(w_j - t_j^2 + 1) + 0.1(w_j - t_j^2 + 1 - 2t_j)$

# 5.4 Runge-Kutta Methods

The goal is to keep the accuracy of Taylor's method (Recall error bound $O(\Delta t^n)$) without computing $f'(t,y), \dots, f^{(n)}(t,y)$

Recall: The IVP, $y' = f(t,y)$;
$(a \le t \le b)(y(a) = \alpha)$

Replace the derivatives
1) Use divided differences / $(n+1)$ pt formula
   $\Rightarrow$ Lagrange interpolant (related to 5.6)
2) Taylor's formula
   ↳ Today (Because $t_j + \Delta t$ and $t_j$ are close)

Problem: $f(t, y(t))$; Need Taylor's formula for 2-var. functions

2-variables functions

$f(x) = ?$

$$f\underset{(t,y_0)}{(x)} = f\underset{(t_0,y_0)}{(x_0)} + \frac{f'\overset{(t_0)}{(x_0)}\overset{(t-t_0)}{(x-x_0)}}{1!} + \frac{f''\overset{(t_0)}{(x_0)}\overset{(t-t_0)^2}{(x-x_0)^2}}{2!} \dots$$

$$f(t,y_0) = f(t_0,y_0) + \frac{\frac{\partial f}{\partial t}(t_0,y_0)(t-t_0)}{1!} + \frac{\frac{\partial^2 f}{\partial t^2}(t_0,y_0)(t-t_0)^2}{2!}$$

$$f(t_0,y) = f(t_0,y_0) + \frac{\partial f}{\partial y}(t_0,y_0)\frac{(y-y_0)}{1!} + \frac{\partial^2 f}{\partial y^2}(t_0,y_0)\frac{(y-y_0)^2}{2!}$$

★ # of partials per derivative

2nd Order

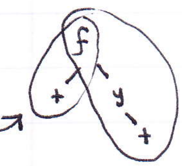
$$f(t,y) = f(t_0,y_0) + \left[\frac{\partial f}{\partial t}(t_0,y_0)(t-t_0) + \frac{\partial f}{\partial y}(t_0,y_0)(y-y_0)\right]$$

$$+ \left[\frac{\partial^2 f}{\partial t^2}(t_0,y_0)\frac{(t-t_0)^2}{2!} + \frac{\partial^2 f}{\partial y^2}(t_0,y_0)\frac{(y-y_0)^2}{2!} + \frac{\partial^2 f}{\partial t\,\partial y}(t_0,y_0)(t-t_0)(y-y_0)\right]$$

Higher-order terms: linear combination of all partials

Goal: Start with $T^{(2)}$ (second order Taylor's method)

- replace derivatives




Recall: $T^{(2)}(t,y) = f(t,y) + \frac{\Delta t}{2} f'(t,y)$

$$= f(t,y) + \frac{\Delta t}{2}\left(\frac{\partial f}{\partial t} + y'(t)\frac{\partial f}{\partial y}\right)$$

$$= \text{GOAL} \rightarrow \quad a f(t+b, y+c)$$

$$= f(t,y) + \frac{\Delta t}{2}\frac{\partial f}{\partial t} + \frac{\Delta t}{2} y' \partial f/\partial y$$

Taylor's formula tells us:

$$a f(t+b, y+c) = a f(t,y) + a\frac{\partial f}{\partial t}(t,y)[\cancel{t}+b-\cancel{t}] + a\frac{\partial f}{\partial y}(t,y)[\cancel{y}+c-\cancel{y}]$$

$$= a f(t,y) + ab\,\partial f/\partial t + ac\,\partial f/\partial y$$

To be equal; $(a=1)\ (b = \Delta t/2)\ (c = \frac{\Delta t}{2} y' = \frac{\Delta t}{2} f(t,y))$

Conclusion:

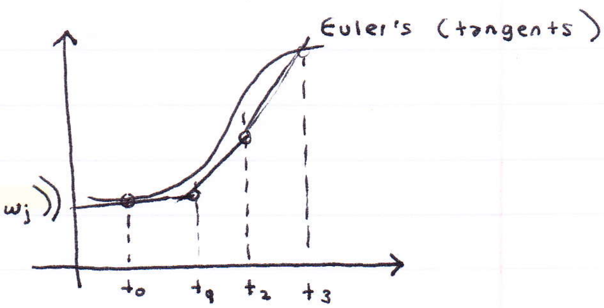
$T^{(2)}(t,y) = f\left(t + \frac{\Delta t}{2}, y + \frac{\Delta t}{2} f(t,y)\right)$ Runge - Kutta method of order 2

$O(\Delta t^2)$ ; (RK 2)

RK2

midpoint method

$$\begin{cases} w_0 = \alpha \\ w_{j+1} = w_j + \Delta t f\left(\left(t_j + \frac{\Delta t}{2}\right), w_j + \frac{\Delta t}{2} f(t_j, w_j)\right) \end{cases}$$




Euler's : $w_{j+1} = w_j + \Delta t f(t_j, w_j)$

$O(\Delta t)$ so RK2 is better

RK2 takes tangent at the midpoint

You can extend this method to get higher order methods

$$\left( \text{(RK4)} \quad O(\Delta t^4) \qquad \begin{aligned} &w_0 = \alpha, \quad k_1 = f(t_j, w_j) \\ &k_2 = f(t_j + \tfrac{\Delta t}{2}, w_j + \tfrac{k_1}{2}) \\ &k_3 = f(t_j + \Delta t/2, w_j + k_2/2) \\ &k_4 = f(t_{j+1}, w_j + k_3) \\ w_{j+1} &= w_j + \tfrac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned} \right)$$

## Modified Euler's $O(\Delta t^2)$

Idea: average of the slopes, at $t_j$ and $t_{j+1}$

$$w_{j+1} = w_j + \frac{\Delta t}{2}\left(f(t_j, w_j) + f(t_{j+1}, w_{j+1})\right)$$

$$= w_j + \frac{\Delta t}{2}\left(f(t_j, w_j) + f(t_{j+1}, \underbrace{w_j + \Delta t\, f(t_j, w_j)}_{w_{j+1} \text{ from Euler's}})\right)$$

Ex) $y' = t^2 + \cos y \qquad 0 \le t \le 1 \qquad y(0) = 1$

1) Well posed? $\quad \partial f/\partial y = 0 + (-\sin y)$

$$|-\sin y| \le L \quad ; \quad |-\sin y| \le 1$$

$1 > 0$ so its well-posed

2) Apply RK2 for general $N$ ; $\quad \Delta t = \frac{1-0}{N} = \frac{1}{N}$ ; $\quad w_0 = 1$

$$w_{j+1} = w_j + \Delta t\, f(t_j + \tfrac{\Delta t}{2}, w_j + \tfrac{\Delta t}{2} f(t_j, w_j))$$

$$t_j = a + j\Delta t = 0 + j/N = j/N$$

$\star\ f(t_j, w_j) = t_j^2 + \cos w_j$

$$w_{j+1} = w_j + \frac{1}{N} f\left(\frac{j}{N} + \frac{1}{2N}, \frac{1}{2N} f\left(\frac{j}{N}, w_j\right)\right)$$

$$w_{j+1} = w_j + \frac{1}{N}\left(\left[\frac{j+1/2}{N}\right]^2 + \cos\left(w_j + \frac{1}{2N}\underbrace{f\left(\frac{j}{N}, w_j\right)}\right)\right)$$

$$w_{j+1} = w_j + \frac{1}{N}\left(\left[\frac{j+1/2}{N}\right]^2 + \cos\left(w_j + \frac{1}{2N}\left(t_j^2 + \cos(w_j)\right)\right)\right)$$

What do Euler's, Taylor's, and Runge-Kutta 2 have in common?

> $w_{j+1}$, depending on $w_j$ (ONE STEP METHOD)

> Multi-Step = get info for $w_{j+1}$ from $w_j$, $w_{j-1}$, ... etc.

# Section 5.6 - Multisteps Method

So far, all methods that we learned are ONE STEP methods. (only $w_j$)

Goal of multi-step: take more information "from past" ($w_j$, $w_{j-1}$, $w_{j-2}$, ...) and also from the future ($w_{j+1}$)

General Expression of a m-step method

$w_0 = \alpha_0, \ldots, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \ldots \quad w_{m-1} = \alpha_{m-1}$

$$w_{j+1} = a_0 w_j + a_1 w_{j-1} + a_2 w_{j-2} + \ldots + a_m w_{j+1-m}$$
$$+ \Delta t\,(b_0 f(t_{j+1}, w_{j+1}) + b_1 f(t_j, w_j) + b_2 f(t_{j-1}, w_{j-1}) + \ldots + b_m f(t_{j+1-m}, w_{j+1-m}))$$

Use a ONE STEP method to calculate these

If $b_0 = 0$, everything only depends on past (EXPLICIT)

If $b_0 \neq 0$, $w_{j+1}$ on both sides (IMPLICIT)

How to get these mult-step methods?

Recall that $y' = f(t, y)$ $\quad y(a) = \alpha$

$$\int_{t_j}^{t_{j+1}} y'(t)\,dt = \int_{t_j}^{t_{j+1}} f(t, y(t))\,dt$$

$\star$ $\int_a^b f'(x)\,dx = f(b) - f(a)$ fundamental theorem of calculus, so

$$\int_{t_j}^{t_{j+1}} y'(t)\,dt \Rightarrow y(t_{j+1}) - y(t_j) = \int_{t_j}^{t_{j+1}} f(t, y(t))\,dt$$

$$y(t_{j+1}) = y(t_j) + \underbrace{\int f(t, y(t)\,dt}$$

Goal: Rewrite as interval; $\int_{t_j}^{t_{j+1}} F(t)\,dt$?

Answer: Lagrange Interpolant $\int_{t_j}^{t_{j+1}} P(t)\,dt$

First Test:

Take only two points to create the Lagrange Interpolant of $f(t, y)$

Here, for an explicit method use $t_j$, $t_{j-1}$

$$P(t) = f(t_{j-1}, y(t_{j-1}))\, L_{1,0}(t) + f(t_j, y(t_j))\, L_{1,1}(t)$$

$$L_{1,0}(t) = \frac{t - t_j}{t_{j-1} - t_j} = \frac{-1}{\Delta t}(t - t_j) \qquad L_{1,1}(t) = \frac{t - t_{j-1}}{t_j - t_{j-1}} = \frac{1}{\Delta t}(t - t_{j-1})$$

$$\int_{t_j}^{t_{j+1}} P(t)\,dt = -\frac{1}{\Delta t}\int_{t_j}^{t_{j+1}} (t - t_j)\,dt + \frac{1}{\Delta t}\int_{t_j}^{t_{j+1}} (t - t_{j-1})\,dt$$

$$\int_{t_j}^{t_{j+1}} (t - t_j)\,dt = \left[\frac{t^2}{2} - t t_j\right]_{t_j}^{t_{j+1}} \qquad \int_{t_j}^{t_{j+1}} (t - t_{j-1})\,dt = \left[\frac{t^2}{2} - t t_{j-1}\right]_{t_j}^{t_{j+1}}$$

$$= -\frac{1}{2\Delta t}\left(t_{j-1}^2 - 2 t_j t_{j-1} + t_j^2\right) \ldots$$

$$\frac{-1}{2\Delta t}(t_{j-1} - t_j)^2 + \frac{1}{2\Delta t}(t_{j+1} - t_j)^2$$

... conclusion?

Conclusion:

$$y(t_{j+1}) = y(t_j) + \int_{t_j}^{t_{j+1}} f(t, y)\,dt$$

$$\approx y(t_j) + \int_{t_j}^{t_{j+1}} P(t)\,dt$$

$$\approx y(t_j) + \frac{\Delta t}{2}\left(3 f(t_j, y(t_j)) - f(t_{j-1}, y(t_{j-1}))\right)$$

$$w_{j+1} = w_j + \frac{\Delta t}{2}\left(3 f(t_j, w_j) - f(t_{j-1}, w_{j-1})\right)$$

This is 2 step Adams-Barshforth Method $O(\Delta t^2)$

| | | |
|---|---|---|
| Explicit Methods) $O(\Delta t^m)$ | + calculus is fast <br> − $\Delta t$ is restricted | Adams-Bashforth |
| Implicit Methods) $O(\Delta t^{m+1})$ | − solve an equation at each step <br> + no time step constraint | Adams-Moutton |

For implicit, we would use $t_j$, $t_{j-1}$, $t_{j+1}$

# Section 5.10: Stability, Consistency, Convergence, and of numerical schemes

Keys in a numerical scheme

1-step VS multi-step

Implicit VS Explicit

Goal of today: determine if our numerical scheme is really approaching the solution of our IVP.

Recall

$$\begin{cases} y' = f(t,y) & a \le t \le b \\ y(a) = \alpha \end{cases}$$

For numerical schemes, we have 2 types of error

- error in the model: we approach the IVP with a numerical scheme (LOCAL Truncation) ($\tau(\Delta t)$)
- error for the solution: discrete VS exact solution ($|w_j - y(t_j)|$)

What about stability?

I) Stability, consistency, convergence for 1-step methods

Recall: For a 1-step method,

$$w_0 = \alpha$$
$$w_{j+1} = w_j + \Delta t\, \phi(t_j, w_j, \Delta t)$$

local trunc. error:

$$\tau_{j+1}(\Delta t) = \frac{y(t_{j+1}) - y(t_j)}{\Delta t} - \phi(t_j, y(t_j), \Delta t)$$

Definition: A 1-step numerical scheme is CONSISTENT if " the local truncation error $\tau_j(\Delta t)$ goes to 0 as $\Delta t$ goes to 0"

aka $\lim\limits_{\Delta t \to 0} \max\limits_{0 \le j \le N} |\tau_j(\Delta t)| = 0$

implies

Definition: A 1-step method is CONVERGENT if " the absolute error of the solutions goes to error as $\Delta t$ goes to 0"

aka $\lim\limits_{\Delta t \to 0} \max\limits_{0 \le j \le N} |w_j - y(t_j)| = 0$

Ex) Show that Euler's method is consistent and convergent

- $w_0 = \alpha$ , $\Delta t = {}^{b-a}/_N$ , $w_{j+1} = w_j + \Delta t\, f(t_j, w_j)$
- $\tau_{j+1}(\Delta t) = \dfrac{w_{j+1} - w_j}{\Delta t} - f(t_j, w_j)$

As $\Delta t \to 0$ ; $\dfrac{y(t_j + \Delta t) - y(t_j)}{\Delta t} \Rightarrow y'(t_j)$ derivative

$T_{j+1}(\Delta t) \to y'(t_j) - f(t_j, w_j)$ ★ $y' = f(t, y)$

so $\tau_{j+1}(\Delta t) = 0$ CONSISTENT

- $|w_j - y(t_j)| \le \underbrace{\Delta t\, M/_{2L}}_{\text{becomes } 0} \left(e^{L(t_j - \alpha)} - 1\right)$ L = lipschitz, M = upper bound

As $\Delta t \to 0$ ; becomes 0 CONVERGENT

→ All 1-step methods we learned are both consistent and convergent

The Stability of 1-Step Methods VS stability for IVP

{ small changes in the initial condition or the function provided a Solution close to the exact solution of IVP. }

VS

{ small perterbation in the initial condition produce small perturbation of the approximation. }

How to ensure that ?

Theorem: If we have $\begin{cases} w_0 = \alpha \\ w_{j+1} = w_j + \Delta t\, \phi(t_j, w_j, \Delta t) \end{cases}$

If $\phi$ is continuous, satisfies a Lipschitz condition,

for small $\Delta t$ → the num. method is STABLE and

consistency ⇔ convergence

II) Stability, consistency, convergence for multi-step methods

Recall: $w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \ldots, \quad w_{m-1} = \alpha_{m-1}$

$$w_{j+1} = a_0 w_j + a_1 w_{j+1} + \cdots + a_{m-1} w_{j+1-m} + \Delta t\, F(t_{i+1}, \ldots t_{i+1-m}, w_{j+1}, \ldots w_{j+1-m})$$

$$0 = \frac{w_{j+1} - (a_0 w_j + \cdots + a_{m-1} w_{j+1-m})}{\Delta t} - F(t_{i+1}, \ldots, t_{i+1-m}, w_{j+1}, \ldots, w_{j+1-m})$$

Definition for CONSISTENCY valid for multi-step using above as $\tau_{j+1}(\Delta t)$

Same for CONVERGENCE

Stability of mult-step methods rely on the roots of the characteristic polynomial

To build the characteristic polynomial:

1) Take the first part of your scheme

$$w_{j+1} = a_0 w_j + a_1 w_{j-1} + \cdots + a_{m-1} w_{j+1-m}$$

2) Replace $w_{j+1}$ by $\lambda^m$, $w_j$ by $\lambda^{m-1}$, $\ldots$ $w_{j+1-m} = 1$

$$P(\lambda) = \lambda^m - a_0 \lambda^{m-1} - \cdots - a_{m-1}$$

Ex) $w_{j+1} = 2w_j + \Delta t\, F(t_j, w_j)$ $\qquad P(\lambda) = \lambda - 2$

Ex) $w_{j+1} = w_j - w_{j-1} + 2(w_j - 8) + \Delta t\, F(t_j, w_j)$

$$P(\lambda) = \lambda^9 - \lambda^8 + \lambda^7 - 2$$

# Sec 5.10 (part II): Stability of Multi-Step methods

★ Characteristic for multi-step methods on 4/5/18 Lecture Notes

$P(\lambda) = \lambda^m - a_0\lambda^{m-1} - a_1\lambda^{m-2} - \dots - a_{m-1}$

Ex) $w_{j+1} = 2w_j - 3w_{j-1} + w_{j-4} + \Delta t\, F(t_{j+1}, \dots, t_{j-4}, \dots, w_{j-4})$

$P(\lambda) = ?$

★ $(m-1 = j-4)$ 5 step method

$P(\lambda) = \lambda^5 - 2\lambda^4 + 3\lambda^3 - 0\lambda^2 - 0\lambda + (-1)$

$\boxed{P(\lambda) = \lambda^5 - 2\lambda^4 + 3\lambda^3 - 1}$

**Goal:** Study the roots of $P(\lambda)$ to determine stability of the scheme

- What is the degree of $P(\lambda)$? ($m^{th}$ degree)

**Definition:** (root condition) Give $P(\lambda)$, a polynomial of degree $m$, and denote by $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_m$ the roots. If all the roots $\lambda_i$ $(i = 1, \dots, m)$ are such that $|\lambda_i| \leq 1$, then $P(\lambda)$ satisfies a root condition

For numerical schemes, if $|\lambda_i| \leq 1$ for the roots of the characteristic polynomial $P(\lambda)$, then we say that the numerical method satisfies the root condition.

| UNSTABLE | STABLE | |
|---|---|---|
| no root condition ↳ there is at least one $\lambda_i$ such that $\lvert\lambda_i\rvert > 1$ | root condition ✓ ↳ $\lvert\lambda_i\rvert \leq 1$ for $i = 1, \dots, m$ | |
| | STRONGLY Stable | WEAKLY Stable |
| | only one $\lvert\,\rvert = 1$ in set | more than one $\lvert\,\rvert = 1$ in set |

Example sets of roots that satisfy root condition

$\{1, -1, 0.5, i, 0.25, -0.99\} \xrightarrow{\lvert\,\rvert} \{1, 1, 0.5, 1, 0.25, 0.99\}$ weakly

$\{1, 0.1, 0.2, 0.99\} \xrightarrow{\lvert\,\rvert} \{1, 0.1, 0.2, 0.99\}$ Strongly

**Theorem:** 1) a multi-step method is STABLE if and only if it satisfies the root condition.

2) if method is CONSISTENT then STABLE $\Longleftrightarrow$ CONGRUENT

Ex) AB2: $w_{j+1} = w_j + \frac{\Delta t}{2}\left(3f(t_j, w_j) - f(t_{j-1}, w_{j-1})\right)$

AM2: $w_{j+1} = w_j + \frac{\Delta t}{2}\left(5f(t_{j+1}, w_{j+1}) + 8f(t_j, w_j) - f(t_{j-1}, w_{j-1})\right)$

Random: $w_{j+1} = 2w_j + \Delta t\left(f(t_j, w_j) - 2f(t_{j+1}, w_{j-1})\right)$ Investigate STABILITY

→ AB2) $P(\lambda) = \lambda^2 - \lambda \Rightarrow \lambda(\lambda - 1)$ ; $\{0, 1\}$ STRONGLY STABLE

→ AM2) $P(\lambda) = \lambda^2 - \lambda \Rightarrow \lambda(\lambda - 1)$ ; $\{0, 1\}$ STRONGLY STABLE

→ Random) $P(\lambda) = \lambda^2 - 2\lambda \Rightarrow \lambda(\lambda - 2)$ ; $\{0, 2\}$ UNSTABLE

# Sec 5.11 - Stiff Differential Equations

<u>Definition:</u> a Stiff Differential Equation is stable (well-posed) for which common time-stepping methods are UNSTABLE, except for small $\Delta t$

How to identify a stiff equation?

→ This is characterized in the solution (of IVP) by a term of the form $\boxed{e^{-ct}}$ $c > 0$, (large)

Ex) $y(t) = e^{-20t}$ ✓ , $y(t) = \frac{e^{-0.5t}\sin(t^2)}{\sqrt{t^2+1}}$ ? , $y(t) = \sin(t^2)\sqrt{t^2+1}$ X

In general, we test the methods on a simple stiff diff. eqn. to find the restrictions on $\Delta t$.

$y' = -cy$ , $(0 \le t < \infty)$ $y(0) = 1$ ; $y(t) = e^{-ct}$

$f(t,y) = -cy$

Let's apply Euler's $(\Delta t = {}^{T-0}/_N)$

$w_0 = 1$ $\quad w_{j+1} = w_j + \Delta t\, f(t_j, w_j)$

$= w_j + \Delta t(-c w_j)$

$\boxed{w_{j+1} = (1 - c\Delta t) w_j}$ Geometric series

$w_j = (1 - c\Delta t)^j w_0$ $\quad |w_j - y(t_j)| = |(1-c\Delta t)^j - e^{-cj\Delta t}| \xrightarrow{\Delta t \to 0} 0$ ?

$\downarrow \Delta t \to 0$ $\quad \downarrow \Delta t \to 0$

$1 \quad 1$ ✓

★ $|(1-c\Delta t)^j - e^{-cj\Delta t}| \xrightarrow{j\to\infty}$ ?

$\downarrow j\to\infty$ $\quad \downarrow j\to\infty$

$0$? $\quad 0$

Yes, if $|1 - c\Delta t| < 1$

$-1 < 1 - c\Delta t < 1$

$-2 < -c\Delta t < 0$

$\boxed{\frac{2}{c} > \Delta t > 0}$ time restriction

For a multi-step method

Ex) AB2: $w_{j+1} = w_j + \frac{3\Delta t}{2} \underbrace{f(t_j, w_j)}_{= -cy} - \frac{\Delta t}{2} \underbrace{f(t_{j-1}, w_{j-1})}_{= -cy}$

$(y' = -cy)$

$= \boxed{w_j \left(1 - c\,{}^{3\Delta t}/_{2}\right) + \frac{c\Delta t}{2} w_{j-1}}$ example

$\downarrow$

Do characteristic polynomial:

$\boxed{P(\lambda) = \lambda^2 - \left(1 - {}^{c3\Delta t}/_{2}\right)\lambda - {}^{c\Delta t}/_{2}}$

$\hookrightarrow$ root condition, on $\Delta t$